Teaching and Learning of Geometry 3-19

Report for the Joint Mathematical Council of the UK

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June 2026

Acknowledgements

We should like to extend thanks to the JMC trustees – first, for giving us the opportunity to develop the interesting piece of work represented in this report, and second, for the care and attention they gave to the draft version, resulting in significant strengthening of our initial document.

Contents

Executive Summary

This report reviews the current evidence on teaching and learning geometry in the UK up to the age of 19, updating the 2001 Joint Mathematical Council/ Royal Society report on teaching and learning geometry. The 2001 report set out eight 'principles' and 16 'recommendations' for teaching and learning geometry for 11–19-year-olds. It suggested that geometry should be a significant part of the mathematics curriculum for all students in this age group, and emphasised the importance of developing an effective teaching model. While most of the 2001 report's conclusions remain valid, we consider it particularly urgent to make more explicit statements about geometrical reasoning in the current curriculums and to change assessment and exam questions related to geometry.

Key conclusions

The teaching and learning of geometrical reasoning are central: While the recognition and identification of shapes and positions are essential, research evidence repeatedly suggests that the most important element of the teaching of geometry is to develop geometrical reasoning. This includes the development of spatial reasoning throughout schooling, but in particular in the early years of education (e.g. ages 3-7). More explicit statements and emphasis on geometrical reasoning are necessary to make the current teaching of geometry more meaningful and enriching for learners.

The current assessment and exam questions provide limited opportunities for using and applying geometrical reasoning: The current assessment and exam questions are mainly for a quick recall of known facts and producing well-rehearsed methods. More careful questioning of assessment specifications and introduction of multi-part questions are needed, e.g., retaining the part that requires factual recall, but then using that same procedural knowledge to demonstrate reasoning and conceptual understanding.

There are different forms of geometrical reasoning and competencies, which need to be carefully taught and developed in the geometry curriculum 3-19: Learners at different stages of school use different forms of geometrical reasoning, such as visual, property-based or deductive reasoning. To make effective use of geometrical reasoning, in addition to securing sound knowledge of geometrical concepts, competencies such as constructing arguments in geometry, interpreting and constructing diagrams and definitions, and spatial skills should be developed throughout the 3–19 geometry curriculum, giving careful consideration to progression and task design. The current curriculums and schemes offer limited scope for geometry, particularly with regard to using and applying geometrical reasoning.

Technological tools such as Dynamic Geometry Software are useful tools. When used appropriately they can help learners to develop their geometrical reasoning and competencies: When the 2001 report was written use of digital technologies was limited. Since then, we have access to many more tools such as GeoGebra, Virtual Reality, and AI. However, just having access does not equate to good or appropriate use, and does not necessarily lead to effective teaching and learning, nor support students' geometrical reasoning and problem-solving skills. Access to and effective use of such tools is still limited.

For geometry teaching to be a rich learning experience ITE and CPD opportunities need to be carefully planned and delivered: Evidence suggests that some teachers lack the confidence to deliver effective geometrical reasoning lessons. University mathematics ITE specialists should work alongside teachers and support them with developing the competencies and designing geometrical reasoning tasks.

Recommendations

This set of recommendations calls for a shift in geometry education towards reasoning, problem-solving, and future-oriented teaching practices.

Recommendation 1: Change the curriculum and assessment.

1.1: Greater emphasis should be placed on geometrical reasoning including visualisation, spatial and problem-solving skills, moving beyond memorising procedures and formulas. Students should be encouraged to explore patterns, formulate conjectures, seek solutions and exercise their spatial reasoning skills. Geometrical reasoning and problem-solving should be embedded explicitly within the 3-19 curriculum.

1.2: The use of dynamic geometry environments and digital technologies, including AI, should be integrated to support conceptual understanding and new ways of thinking. This requires a shift in teacher beliefs and practices, supported by appropriate training.

1.3: Assessment should provide not only procedural testing but also include multi-part questions that begin with factual recall and lead to geometrical reasoning, including spatial skills, and problem-solving.

Recommendation 2: Focus on ITE/CPD

2.1: There is a need to recruit and retain more subject specialists, alongside better support for non-specialist teachers, particularly where they are teaching lower years and lower attainment groups.

2.2: Teachers require high-quality CPD that provides effective support for geometrical reasoning, task design, and use of digital technologies, including AI, ensuring these tools are used to enhance, rather than replace, learning.

2.3: Teachers should be supported as researchers. Stronger links between schools and universities are needed, with beginning teachers paired with mentors engaged in practitioner research.

Recommendation 3: Funding localised classroom-based research

3.1: Teachers should be supported to design rich geometrical tasks grounded in sound pedagogy, real-world contexts, and the cultural and historical dimensions of mathematics. Professional development should support both student engagement and teachers' own understanding through the creation of such tasks.

3.2: More funding for conducting classroom-based research, e.g., how to develop geometrical reasoning, spatial skills and their competencies for the era of AI. Such funding does not have to be huge, but enough for both researchers and teachers to work together to conduct their studies, distributed widely across the UK (and internationally).

1. Introduction

> So geometry is an important subject, with wide applications and a long history. It deals with matters we find attractive and for which we have a strong visual capacity. On the surface, then, it would appear that geometry should be one of the easiest branches of mathematics to teach. But this is not the case - neither in England nor in much of the developed world. This Royal Society / JMC study set out to identify why this is so. (JMC/RS, 2001, p. vii)

In 2001, the Joint Mathematical Council and Royal Society (JMC/RS) published a report on the teaching and learning of geometry 11-19. This seminal work identified the eight ‘principles’ and 16 ‘recommendations’. For example, Principle 1 addresses the place of geometry in the mathematics curriculum, while Principle 7 focuses on pedagogy. The report concluded that ‘Bringing about improvements in geometry teaching will require a significant commitment to a substantial programme of continuing professional development, together with the development of appropriate supporting materials’ (p. 23). Since then, efforts have been made by ‘committed’ people to establish the role of geometry in the mathematics curriculum and to develop effective teaching methods. For example, the Geometry Working Group of the British Society for Research into Learning Mathematics (BSRLM) presented a variety of ideas and recommendations based on their research (e.g., Jones and Fujita, 2001). By contrast, Jones and Mooney (2003) observed that topics related to geometry were quite limited compared to those related to number and algebra, while the JMC/RS Geometry Report 11-19 (2001, hereinafter the 2001 report) recommended, for example, ‘Geometry should form a significant component of the mathematics curriculum for all students from 11 to 19’. Although the National Curriculum in 2014 in England did include ‘Measurement’ and ‘Geometry’ (properties of shapes and position/direction), Ofsted in 2023 observed that there might be crucial issues in the teaching and learning of geometry, stating:

> A small proportion of learners [in primary schools] were relatively insecure in their knowledge of geometry facts. This was possibly because geometry topics had been allocated to the summer term, leading to long gaps before learners revisited them.

Reflecting on the 2001 report, curriculum developments and research studies on geometry, the aim of this report is to provide an updated, research-informed account of the teaching and learning of geometry within the current educational and technological landscape. Specifically, Section 2 of this report critically examines the eight ‘principles’ and 16 ‘recommendations’, building on developments in geometry education research and classroom practice since 2001. In Section 3, we discuss the teaching of geometry 3-19. In Section 4 we set out revised recommendations relevant to the current curriculums, policy makers, and schools in the UK.

2. Evaluation of the key principles and recommendations from the 2001 report

The Key Principles (P1-P8) and Recommendations (R1-R16) of the 2001 report are broadly separated into three themes (for each Principle and Recommendation, please see Appendix 1 in this report):

- Theme 1: The status of geometry in the mathematics curriculum [P1, P4, P5, P8; R1, R2, R4, R9, R10, R11, R12, R13, R16]
- Theme 2: Pedagogy underpinning the teaching and learning of geometry [P2, P7; R3, R5, R15]
- Theme 3: Geometry curriculum and assessment [P3, P6, P7, P8; R3, R4, R5, R6, R7, R8, R13, R14]

Theme 1: The status of Geometry in the mathematics curriculum

The recommendations R1, R10, R11 and R12 in the 2001 report argued that geometry should form a 'significant component of the mathematics curriculum for all students from 11 to 19'. Examining the current national curriculum, and existing schemes, our observation is that the 2001 recommendation has only been partially met. For example, in 2014 the National Curriculum (NC) in England was restructured and the use of 'geometry' and 'geometrical thinking' replaced 'Shape, Space and Measures' (R2, R9, R13). This increased the curriculum alignment with East Asian countries, where formal terminology is used in order to encourage mathematical rigour (Forestier and Crossley 2015; Liu, 2018).

We examined some widely used mathematics schemes and noted that 11-16-year-old learners in England were allocated about 20% of the curriculum time for geometry. However, in primary school, it was only 12-14% of the curriculum time, and often at the end of the school year. In Scotland (Curriculum for Excellence, CfE, 2017) and Northern Ireland (Council for Curriculum, Examinations and Assessment, CCEA, 2019) students follow a similar programme of study. Interestingly, both the CfE Benchmarks guidance and CCEA make no mention of 'geometry'. Instead, the CfE refers to 'Shape, Positions, and Movement', and CCEA refers to 'Shape and Space'. The content covered by the Curriculum for Wales (CfW, 2016) includes 'Using Geometry Skills'. It appears that across the UK the time spent on teaching geometry falls short of the 25-30% recommended by the 2001 report for 11-16-year-olds. Where students study mathematics post-16, the A-level syllabus often only covers coordinate geometry (AQA, Edexcel). As a consequence, not much time is given to teaching and learning geometry at a higher level, and there is little opportunity for learners to develop their geometrical knowledge and spatial skills (Ofsted 2023), nor find geometry to be rich and rewarding (as pointed out by Jones and Mooney, 2003).

Overall, the 11-16 curriculums across the UK give similar status to teaching geometry, but there is insufficient emphasis in the 3-11 and post-16 curriculums.

Theme 2: Pedagogy underpinning the teaching and learning of Geometry

One of the strong statements of the 2001 report was key principle 7, which stated that 'the most significant contribution to improvements in geometry teaching will be made by the development of good models of pedagogy'. Echoing this statement, Jones and Fujita (2001) identified suggestions for developing geometry pedagogy, including ways to enhance students' geometrical intuition, teach 3D geometry and utilise dynamic geometry software for geometrical reasoning. Reviewing research literature, it seems that the international mathematics education communities have developed many useful ideas and examples for good models of pedagogy, e.g., practical and experimental, modelling approaches, the use of technological tools, cultural-social and ecological, and so on (e.g., Fujita and Jones, 2003; Rowlands, 2010; Sinclair et al., 2016; Weigand et al., 2025). As a pedagogical approach, Seah and Horne (2021) suggested that to develop geometric reasoning, it is necessary to provide opportunities for students to a) discuss their understanding, b) use visual representations to support their reasoning, c) move between different representations, and d) learn the language of explanation, argument and justification.

Meanwhile, UK mathematics education has mainly focused on developing numeracy-related pedagogies. For example, in England, the introduction of the National Numeracy Strategy in 1999 inspired research studies related to pedagogies for teaching number and calculation, but not much about geometry (Jones and Mooney, 2003). Since 2014, Teaching for Mastery (TfM) has been a key pedagogy underpinning teaching and learning within the National Curriculum for England. TfM is defined as acquiring 'a deep, long-term, secure and adaptable understanding' through interaction with the four main components: mathematical representation and structure; mathematical thinking; variation; and fluency (NCETM, 2019).

Mastery is also presented as a progression from Type 1 knowledge to Type 2 knowledge:

- Declarative: 'I know that' (type 1 - facts and formulae) to relationships between facts (type 2),
- Procedural: 'I know how' (type 1 - methods) to relationships between facts, procedures and missing facts (type 2),
- Conditional: 'I know when' (type 1 - strategies) to relationship between information, strategies and missing information type 2). (Ofsted, 2021)

What teaching approaches, underpinned by pedagogical ideas suggested by research in geometry teaching could have been used to promote such progression in geometry? Reviewing the current schemes, we speculate that learners might not be required to use their geometrical reasoning, because the tasks they are presented with in class only require procedural knowledge. For instance, tasks in KS2 tests taken by 11-year-olds (Year 6) can often be completed by mainly using known facts about angles at a point on a straight line, such as:

> Four angles are at a point on a straight line, and one angle is 81 degrees. The other three angles are equal. What size are the other three angles?

or

> Calculate the size of interior and exterior angles in a triangle.

This may be a result of primary teachers having insufficient depth of understanding of geometrical reasoning (van Putten et al., 2022), and focusing too closely on assessment questions.

Ofsted points out that reasoning and problem-solving skills are not generic, and learners cannot become problem-solvers by imitating their teachers (Ofsted, 2021). Learners need more mathematically rich tasks that require not only type 1 knowledge but also type 2. During classroom observations, Ofsted noted that there might not be a common understanding of how to teach problem-solving effectively. For example, in primary schools, TfM is often understood as the use of Concrete-Pictorial-Abstract (CPA) modes of representation (Bruner, 1966; NCETM, 2024), but there are a few issues related to geometry. For example, based on anecdotal evidence, we speculate that the approach might be mainly understood as the pedagogy for teaching number and calculation, and there might be a tendency for primary teachers to use too many concrete and pictorial demonstrations, and multiple representations could cause confusion.

Frequently, the Abstract stage was omitted because not enough time was allocated to each topic, and the reasoning tasks were omitted despite the fact that reasoning and problem-solving are key features of the curriculum. The converse is true in secondary schools, where teachers move straight to the abstract stage in an attempt to complete the syllabus (Ofsted, 2023; Leong et al., 2015). One result of this strategy is that secondary learners are often good at examination questions as a result of a narrowed curriculum, but lack a secure understanding for success later. This is often driven by school accountability and the need for learners to achieve threshold grades. Ofsted (2023) goes on to point out that this ‘approach reduces mathematics education to an exercise in preparing learners to pass an external examination and does not provide learners with a rich mathematical education’, instead they view geometry as a collection of unconnected ‘tricks’ to memorise.

Similar themes to TfM are addressed in Scotland’s Curriculum for Excellence (e.g., ‘Multiplying skills, adding value’ report), and in the Curriculum for Wales, which has a focus on quantitative literacy (Forsythe and Smith, 2020).

Therefore, although the recommended pedagogy within the NC, CfE, and CfW may be consistent with the key principles and recommendations within the 2001 report, in practice they are not always met nor addressed (Alderton and Pratt, 2025; Brown, 2011; Resnick et al., 2020; Ofsted, 2021; Ofsted, 2023).

Theme 3: Geometry curriculum and assessment

In the 2001 report, key principle 6 states that ‘The assessment framework for the curriculum should be designed to ensure that the full range of students’ geometrical knowledge, skills and understanding are given credit’. This recommends that geometry should be well established in schools, and provide opportunities for learners to engage in 2D and 3D geometry, so that they develop a good understanding of the properties of shapes, and that assessment should reflect such learning opportunities.

Many schools use commercial schemes to address the difficulties of recruiting and retaining mathematics specialist teachers. These commercial schemes reflect the imbalance in the curriculum noted in Theme 1. We notice that the tasks within these schemes are designed to prepare learners for external examination (e.g. KS2 tests, GCSE in England, Wales and Norther Ireland, National 5s in Scotland) questions, and end up being more procedural rather than conceptual/abstract. The Scottish inspectorate (HMEI, 2024) found that ‘too much emphasis is placed on procedural knowledge and factual recall’.

Consider the similarity of classroom tasks to assessment questions (Appendix 2). The examples show typical geometry questions in the KS2 tests in England’s primary schools. Very little problem-solving or reasoning is required to answer them correctly. Learners are expected to know angle facts (vertically opposite angles, the angle sum of a quadrilateral, angles at a point on a straight line), and are simply required to carry out a relatively straightforward calculation.

A similar comparison of classroom questions with those from Foundation and Higher GCSE calculator papers shows that very little reasoning or problem-solving is required (Appendix 3). All the questions can be completed by using known facts (angles at a point and trigonometric ratios).

Much of the final year of compulsory secondary schooling is spent on exam preparation, and learners spend most of their time revising and doing exam questions. Both Ofsted (2023) and HMEI (2024) note that this reduces their experience to an exercise in passing exams, rather than engaging in something rich and rewarding.

Even for year groups not taking a formal examination, teachers, with their best intentions, tend to use exam questions as a basis for their planning (DfE, 2023; Whetton, 2009; Ro, 2019). Popham (2003) calls this ‘teaching to the test’, and given the high-stakes nature of school inspections, and examination results (which may impact on the next steps students will take, such as university courses), it is easy to see why teachers resort to this form of classroom practice. The problem with ‘teaching to the test’ is that it does not assess or reward geometrical reasoning, or a deep conceptual understanding (Hoyles and Healy, 2007). Instead, it encourages a quick recall of known facts and produces well-rehearsed methods. One way to address this issue without a radical change of the examination system would be to introduce multi-part questions. That is, retain the part that requires factual recall, but then use that same procedural knowledge to demonstrate reasoning and conceptual understanding.

3. The teaching of geometry 3-19

3.1 Our definition of geometry in schools and why geometry is needed

While giving a definition of geometry in schools is very difficult (or impossible), geometry in schools is mainly concerned with shapes and their relationships in 3D Euclidean space. There are good reasons why 3D Euclidean space is used. Freudenthal (1971) once stated, ‘nobody can deny that we live in space, that we move in space, that we analyse space, to be better adapted to it. In fact, we are so used to space that we easily forget about its importance for ourselves and for those we are educating’ (p. 418). Through studying about geometrical space, we expect that students learn various shapes, facts and properties as well as reasoning and problem-solving (Fujita et al., 2025).

The recent PISA framework views geometry from a wider perspective. ‘Space and Shape’ includes explorations of ‘a wide range of phenomena that are encountered everywhere in our visual and physical world: patterns, properties of objects, positions and orientations, representations of objects, decoding and encoding of visual information, navigation and dynamic interaction with real shapes as well as with representations, movement, displacement, and the ability to anticipate actions in space’ (OECD, 2022). The framework goes beyond the study of static shapes, and includes measurement, spatial visualisation and algebra.

Given the perennial quizzing by generations of learners asking their teacher, ‘When are we ever going to use circle theorems?’ We ask ourselves whether geometry still has a place in the school curriculum. Has the prevalence of AI and other technological advances made geometry a redundant topic area?

We respond to this by saying that mathematics in its broadest sense provides learners with opportunities to work with tasks that enable them to develop their reasoning and problem-solving skills, and that geometry in particular provides an ideal context for students to engage with spatial and visual proof, and the formal language needed to think like a mathematician in producing a reasoned, coherent argument (Rowlands and Carson, 2006; Polya, 1945; Lakatos, 1976; van Heile, 1986).

The school mathematics curriculum is, for example, tasked with teaching learners how to ‘reason mathematically by following a line of enquiry, conjecturing relationships and generalisations, and developing an argument, justification or proof using mathematical language’ (NC, 2014). No other aspect of the school curriculum provides learners with a place to engage in spatial and visual reasoning other than geometry (Atiyah,1982).

Geometry also provides a direct path to our cultural background. This cultural and historical thread connects us all, and for that reason alone, we argue that geometry should be a key feature of the school curriculum. ‘Seeing how the subject was built up from its naturalistic origins through cultural historical processes of development is the best way to give students access to those intuitions and understandings’ (Rowlands and Carson, 2024).

3.2 The importance of geometrical reasoning

As set out in the previous section, geometry is not limited to the study of static geometrical shapes, but seeing them dynamically and spatially, and exploring patterns, properties of objects, positions and orientations, representations of objects, decoding and encoding of visual information, navigation and dynamic interaction with real shapes as well as with representations, movement, displacement, and the ability to anticipate actions in space. The central premise of such exploration is geometrical reasoning.

In 1998, Duval claimed that ‘much research about the deep process of the development and the learning of visualisation and reasoning is still needed’. Research has shown that geometrical reasoning has been the main interest in the teaching and learning of geometry. Reflecting on the current status of geometry in schools, we consider that more explicit objectives related to geometrical reasoning in the National Curriculum and associated assessment is essential, and this will make geometry-related topics richer and more interesting. We are not saying other areas of mathematics are not important, but as a starting point, we can start by re-considering our geometry curriculum from a geometrical reasoning point of view.

Different forms of geometric reasoning can be considered. Drawing on previous studies such as van Hiele (1986), Clements and Battista (1992) and Sfard (2008), Seah and Horne (2021) argue that reasoning can be captured by observing when students explain their thinking, particularly when they combine multiple representations, visualisation and mathematical discourse (p. 251). Seah and Horne (2019) also list five types of geometric thinking: visual, descriptive, analytical, relational-inferential property-based, and formal proof (pp. 167-169), which correspond to van Hiele's levels of geometric thinking. According to Seah and Horne (2021), when reasoning visually students mostly use physical properties of shapes. Whilst some students begin to pay informal attention to properties of shapes (descriptive reasoning), other students begin to use relational-inferential property-based reasoning, where they can identify and use necessary geometric properties or facts to reason with more formal language and definitions. Formal proofs are based on the use of definitions, premises and conclusions with inference rules such as universal instantiation, hypothetical syllogism and so on (Miyazaki et al., 2017).

In secondary school, we expect students to learn and use more property-based reasoning (e.g. linking properties of shapes with their definitions) or formal proofs in geometry, although many research studies have evidenced that this might not be so easy (e.g., Jones and Tzekaki, 2016). In primary schools, many students are expected to engage in visual or analytical reasoning (e.g. using properties of shapes with informal language), with appropriate representations.

Geometrical reasoning should be seen not only in two-dimensional but also in three-dimensional spaces. This is not particularly new. For example, in the early 20th Century, a German mathematics educator, Treutlein, who organised intuitive geometry instruction, stated that ‘Spatial considerations in plane geometry should not merely be permitted; they should, in fact, be actively encouraged in the classroom’ (Treutlein, 1911; see also Yamamoto, 2006). Currently, cognitive science and mathematics education research suggest the following skills (Pittalis and Christou, 2010; Lowrie et al., 2018; Fujita et al., 2022, p. 439) are particularly related to spatial thinking and problem-solving:

- Mental rotation - the ability to accurately rotate a 2D shape or a 3D object in the mind’s eye in order to perform a subsequent task.
- Spatial orientation - the ability to perform an egocentric transformation of imagining a change in one’s own perspective.
- Spatial visualisation - the ability to mentally transform or manipulate the visuospatial properties of an object, distinct from rotation of the object (i.e., mental rotation) or varying one's perspective (i.e., spatial orientation).

For example, asking young learners to visualise and sketch a block shape that is described to them, or to imagine rotating a block shape (Schmitt et al., 2018). Fujita et al. (2020) reported that many 12-year-old students’ reasoning about 2D representations of 3D shapes was influenced by the visual appearance of the 2D representations, but when they worked with different representations of 3D shapes, some of them began to use properties of the shapes to make sense of their thinking, utilising spatial skills such as mental rotations or spatial visualisations.

Figure 1 represents a ‘general guidance’ of the progression of geometrical reasoning, indicating expected reasoning types in different stages of school. Of course, this does not mean EYFS/KS1 learners cannot/should not use analytical or property-based reasoning or vice versa. For example, Sinclair and Moss (2012) found that within technology-rich learning environments, young children’s development of geometric discourse can be a patchy, oscillating process in which various forms of geometric reasoning are involved. Furthermore, visualisation is a useful way of attacking challenging questions and problem-solving for older learners, and, visualising shapes and seeing them dynamically or rotating shapes mentally will be very useful in geometry and beyond.

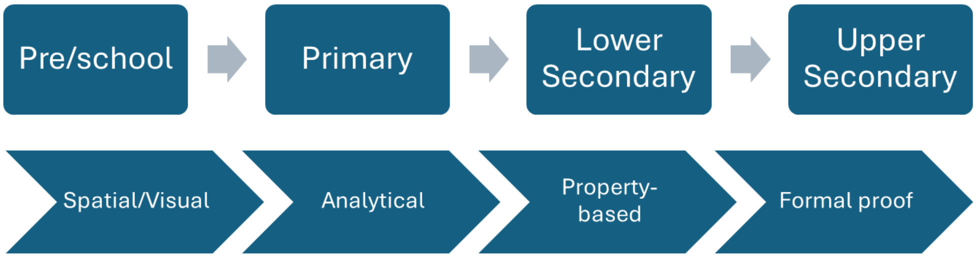

Figure 1. A simplified progression in geometrical reasoning for different stages of school

3.3 Task design for effective teaching of geometry/geometrical reasoning 3-19

The 2001 report specifies the importance of pedagogy in Key Principle 7. While different teaching approaches, ideas and the use of technological tools have been proposed and advocated. Reflecting on our review of the current situation of the teaching of geometry, we feel more activities for geometrical reasoning and utilising problem-solving strategies are necessary. Designing tasks is central in mathematics education (Wittmann, 1995), but it is not easy.

Atiyah (1982) wrote ‘Broadly speaking I want to suggest that geometry is that part of mathematics in which visual thought is dominant ...’ (p. 183), and geometry related problems for students can start with how we use visual thinking. We explore our thinking and reasoning through various representations. In geometry, these can be internal, such as mental images, or external, such as diagrams in textbooks or on computer screens, as well as physical models (Fischbein, 1993; Mesquita, 1998; Duval, 2006). Learners can capture these representations as 'geometric figures', which are psychological concepts that incorporate visual and theoretical elements. For example, a square can be visually represented by drawings, but it is also a theoretical object - a quadrilateral with four equal sides and angles (Fischbein, 1993; Fujita & Jones, 2007). These two aspects - the visual and the conceptual - can be utilised through modelling with geometric figures and diagrams representing abstract geometric shapes.

For example, Herbst et al. (2017) identified four different contexts for geometrical figures in geometrical spaces, summarised in Table 1.

Table 1: Different contexts of geometrical figures in geometrical spaces

Geometrical spaces	Contexts
Macrospace	1.Using geometrical figures to explore or navigate bigger spaces such as planets in space, modelling parks.
Mesospace	2. Using geometrical figures to capture big objects such as buildings, houses, cars, and so on.
Microspace	3. Using geometrical figures to construct shapes as small objects 4. Using geometrical figures to describe or manipulate small objects

Figure 2 shows an example from a Japanese textbook for 13-14-year-old students modelling a park ride (an object in a mesospace) using a parallelogram, as an introduction to properties of parallelograms and their proofs.

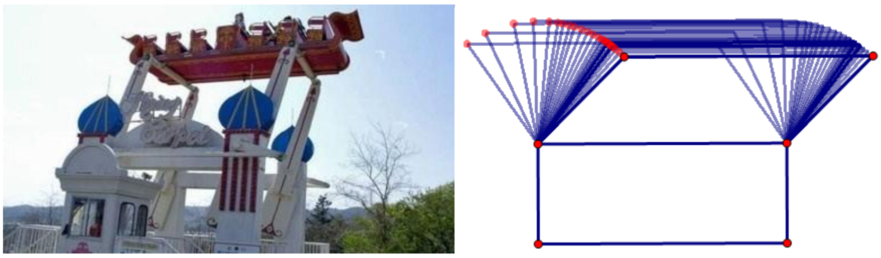

Figure 2. Modelling a park ride (from http://moriumi.web.fc2.com/zhitachi-rejarando.html)

Such modelling activities can also be integrated into the early years of education, where children can connect between micro- and meso- objects that will contribute to developing their dynamic views and spatial reasoning skills (e.g., Marchand et al., 2024).

Another approach for such activities is 'explanatory activity in geometry', where learning opportunities should not be limited to simply knowing the names of certain geometric shapes, but should also provide opportunities for students to explain their thinking about how they knew or noticed something, and why (Fujita et al., 2025). To provide such an opportunity, the following design principles apply:

- The tasks are related to describing why something is happening or being observed, which might be difficult to judge on the basis of measurements/a few examples, or perceptions at a glance. We expect that this could provoke different answers, and class discussions and encourage students to explain each other's ideas.
- The problem can easily be extended to other cases to provide students with coherent learning opportunities.
- Different representations would be useful to ensure that the reason why something is happening is being observed. In particular, manipulating geometric diagrams can be a starting point, including producing counter-examples. Where appropriate, different technological tools can be considered to effectively represent and enable learners to interact with them dynamically.
- Mathematical properties can be identified so that students can use them as necessary narratives to explain why or what they have noticed.

The task in Figure 3 was designed and based on the above principles, requiring learners to develop geometrical explanations, e.g., why triangles 'outside' might be similar to each other (situation 1), or why overlapped areas might always be the same (situation 2). Please see other examples in Appendix 4. These tasks also can be extended to different polygons, e.g. regular hexagons (Fujita et al., 2025).

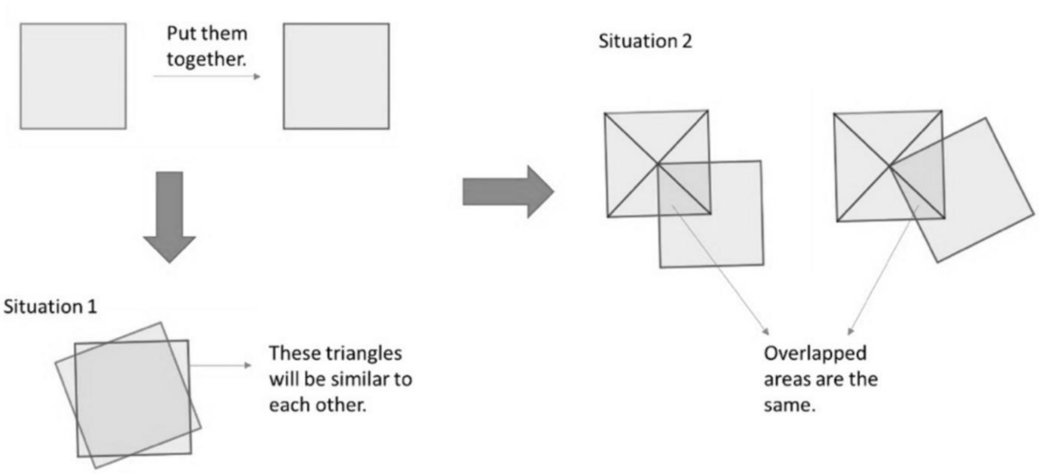

Figure 3. An example of an explanatory activity in geometry

In these tasks, initial exploration allows students to build intuition and discover patterns. However, the task must be structured to leave students with a realisation that their current empirical reasoning might be insufficient to explain why a pattern holds. This might lead to the immediate enquiry while creating the intellectual motivation to seek the next level of geometrical reasoning or formal proofs, e.g. encouraging students to bridge the 'unfilled gap' between initial empirical measurement and formal deduction.

3.4 Evidence of current competencies in geometry: performance, foundations, and affect

What would enable students to utilise geometrical reasoning, and why do some students struggle to do so? Websites such as Third Space Learning (https://thirdspacelearning.com/) repeatedly report that formal examination questions (e.g. KS2 tests, GCSE in England) related to geometry or shapes are harder for many students. To explore these issues, it might be useful to consider what competencies are necessary.

The notion of competencies is a key construct in today's educational research (e.g., Geraniou and Jankvist, 2019). By integrating our evaluation of the 2001 report, existing literature and research findings, we consider the following competencies might be particularly essential to act appropriately in situations involving geometric challenges.

- Competencies related to constructing arguments in geometry
- Competencies related to interpreting information embedded in geometrical diagrams
- Competencies related to spatial skills
- Competencies related to technological tools

While we do not insist that these are the definitive competencies for geometry, we believe these are useful starting points to explore what the teaching of geometry could look like in the next 10 years. Also, considering some knowledge-based curriculum (e.g. English NC), the development of sound knowledge will be essential, and what we are suggesting is not that one takes over the other, but that well-balanced views are necessary.

3.4.1 Competencies related to constructing arguments in geometry

Learners must understand the different types of reasoning, critically examine their respective strengths and weaknesses, and apply them appropriately when constructing arguments to prove geometric theorems. In primary school, for example, we can start from specific measurement values (e.g. 60° + 60° + 60°) to demonstrate that the sum of the interior angles of a triangle is 180°, and then explore if other explanations might offer more general arguments. In secondary school, learners may use properties of angles and parallel lines to reason more deductively. Recommendation 3 of the 2001 report actually suggested the following aims: ‘c) To develop knowledge of, and the ability to use, geometrical properties and theorems. d) To encourage the development of, and the use of, conjecture, deductive reasoning and proof’. However, constructing a deductive argument in geometry is difficult for many students.

Mathematics education research reports that students have difficulty understanding why deductive arguments are required in geometry, and this difficulty has not been fully resolved. For example, Bieda (2010) stated ‘Students, even undergraduate mathematics majors, tend to generate empirically-based justifications instead of constructing deductive proofs and to supplement valid deductive proofs with diagrams or examples’ (p. 352). In fact, it was found that gaps related to the generality of arguments remained unfilled, suggesting that this is one of the most difficult aspects yet to be fully addressed in the teaching of deductive proof in school geometry (Conner, 2022). We need to seek ways to develop learners’ competencies to evaluate different geometrical arguments based on different reasoning.

In a UK study, Hoyles and Kuchemann (2000) found that Year 8 students could identify false geometric statements, such as 'Whatever quadrilateral I draw with corners on a circle, the diagonals will always cross at the centre of the circle', and provide counterexamples or analytical reasoning. They also found that almost half of the students agreed with the false statement and that fewer students provided analytical reasoning.

Learners also have to learn how to construct mathematical argument. Cirillo and Hummer (2021) recommend the following competencies are explicitly taught or developed:

- How to draw conclusions from ‘given’ information
- How to support claims with warrants, knowing the kinds of warrants that are allowable in a proof and understanding their differences
- How to logically proceed through a proof using one or more chains of reasoning, beginning with the ‘given’ information and ending with the ‘prove’ statement
- How to attend to important details in a proof, such as how to write up common sub-arguments

Considering the current curriculums and students’ performance in reasoning, it is important to continue to seek ways to improve their competencies for productive mathematical arguments.

3.4.2 Competencies related to interpreting information embedded in geometrical diagrams

Diagrams play important roles in geometry. Mason (1988) suggests that a diagram ‘can act as scaffolding or support for the mental screen, stabilising the image, but if the diagram becomes the sole object of attention then it can hinder rather than help thinking’ (p. 297). In fact, existing studies have suggested that students demonstrate different ways of drawing and interpreting such diagrams and using the information in geometrical reasoning. In particular, a key competency is related to working with diagrams, critically seeing how these diagrams look (visual) as well as how these diagrams might be defined (conceptual), having and utilising their figural concepts (Fischbein, 1993).

Let us consider the following problems:

> Is it possible to draw a parallelogram whose four vertices are on the circumference of a circle?
>
> Name the shape described by the following statement – 'A parallelogram with one vertex as a right angle'

These questions look simple (for each problem the solution is a rectangle), but the issue related to interpreting diagrams is a persistent one. For example, 40% of Japanese 14-year-old students who study geometrical proofs extensively in their mathematics curriculum could not answer the first question. It is speculated that these students have their prototypical examples of parallelogram (e.g. 'slanted rectangle') even though some of them could state a correct definition of a parallelogram, and often drew the following (Figure 4):

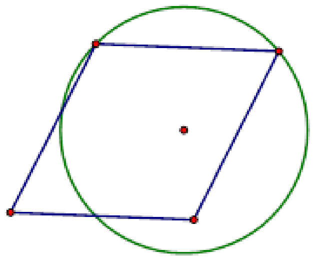

Figure 4. A student's response to drawing a parallelogram in a circle

A similar tendency was also observed in England (Fujita, 2012). In other words, these students might have 'knowledge' of parallelograms (type 1), but this did not function as type 2 knowledge. That is why it is very important to develop student competencies related to interpreting information embedded in geometrical diagrams , harmonising visual and conceptual elements of geometrical diagrams.

Resources such as those from NCETM include some interesting tasks e.g., 'Give children two identical shapes, like these two isosceles triangles. Ask the children to put the shapes together to fit certain criteria, e.g. 'Join these two isosceles triangles to make a parallelogram. Join these two isosceles triangles together to make a quadrilateral with no parallel sides.''(Y3) However, questions related to geometry in KS2 tests provide a limited opportunity to utilise competencies in interpreting geometrical shapes and diagrams as their representations, see the example in Figure 5.

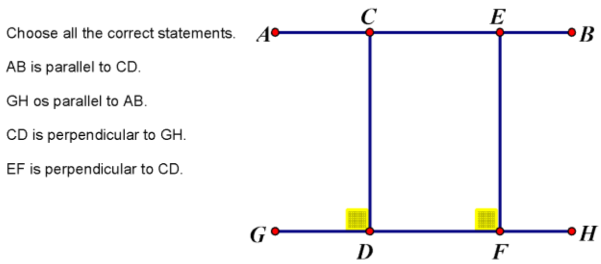

Figure 5. An example of a KS2 test question in 2024 that assesses Type 1 knowledge

3.4.3 Competencies related to spatial skills

The 2001 report suggested one of the aims of geometry education should be 'a) to develop spatial awareness, geometrical intuition and the ability to visualise'. In relation to this, and extending the competencies related to interpreting information embedded in geometrical diagrams, in recent years the importance of spatial reasoning skills has been increasingly recognised (e.g. Lowrie et al., 2021). Spatial reasoning skills are related to other mathematical areas such as number and algebra (Lowrie et al., 2019) as well as STEM subject domains including computer science (Wai et al., 2009; Gifford et al., 2024). The main competencies are, as we have seen, mental rotation, spatial orientation, and spatial visualisation. These skills, together with knowledge in geometry, enable to learners to see geometrical shapes dynamically, and are essential for solving geometry problems with 3D shapes. For example, consider a problem 'Find the size of the angle DEG in a cube, represented in Figure 6.' Although this problem looks simple to solve, studies repeatedly report that many students struggle to provide a correct answer (Fujita, et al., 2020; 2022). Using spatial reasoning skills, this problem can be easily solved.

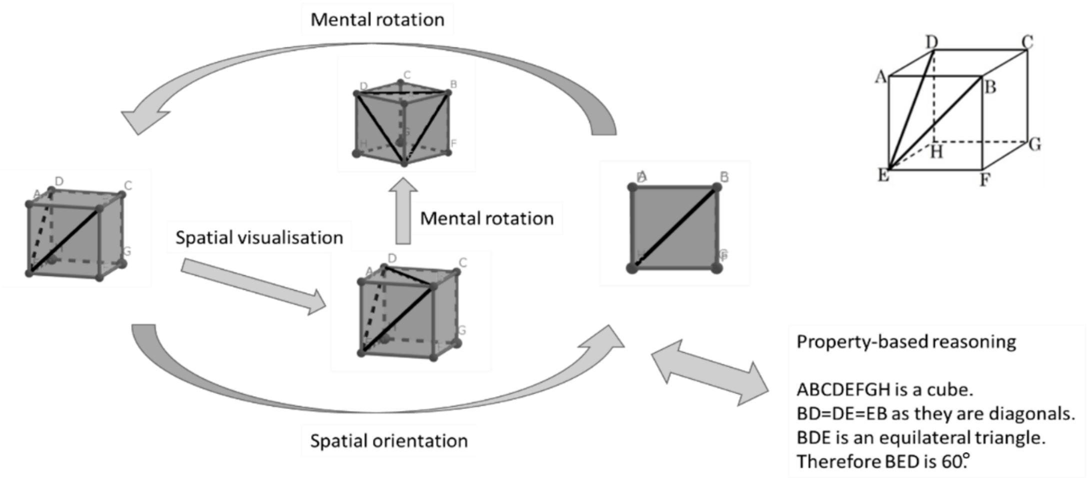

Figure 6. The size of the angle DEG and the related spatial skills

Conversely, we need to reflect on our current practice and check whether we have provided opportunities for learners such that they can develop and exercise their spatial reasoning skills in geometry. In fact, the current exam questions might again provide limited opportunities for spatial reasoning skills (e.g., Figure 7).

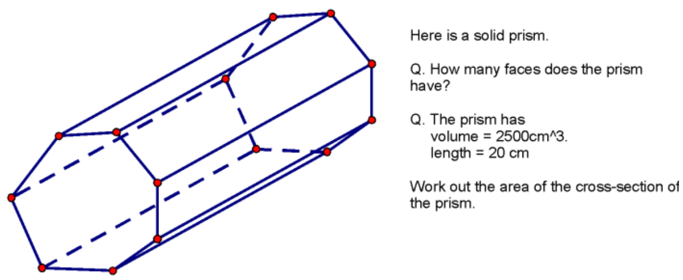

Figure 7. What skills will be useful to solve this question?!

As the Early Childhood Maths Group toolkit suggests, the competencies related to spatial reasoning are crucial in the early years, the skills should be explicitly developed as early as possible through mathematically authentic activities (e.g., Oughton et al., 2024). Activities such as drawing patterns, growing patterns, the use of the programmable toys are effective ways to improve spatial reasoning (Mulligan et al., 2020). The Royal Society ACME's primary and early years expert panel (Gifford et al., 2024) recommends 'spatialising' the mathematics curriculum. It is encouraging to see that the current Early Years Foundation Stage (EYFS) statutory framework (2025) explicitly states 'it is important that the curriculum includes rich opportunities for children to develop their spatial reasoning skills across all areas of mathematics, including shape, space and measures'. Inspiring ideas and examples can be found on the Early Childhood Maths Group's website (https://earlymaths.org/spatial-reasoning/, which includes many activities for spatial reasoning. However, Bates et al. (2023) reported early years practitioners had little confidence in their understanding of spatial reasoning. More studies will need to be conducted about how such skills can be integrated into the early years' curriculum, and how to prepare prospective teachers to have confidence and both subject and pedagogical content knowledge for spatial reasoning in the EYFS.

Finally, considering the recent research findings in spatial reasoning, which are also related to general mathematical competencies (e.g. Resnick, et al., 2020), these skills are related to not only reasoning with 3D shapes but also to problem-solving with 2D shapes. For example, consider the GCSE question in Figure 8.

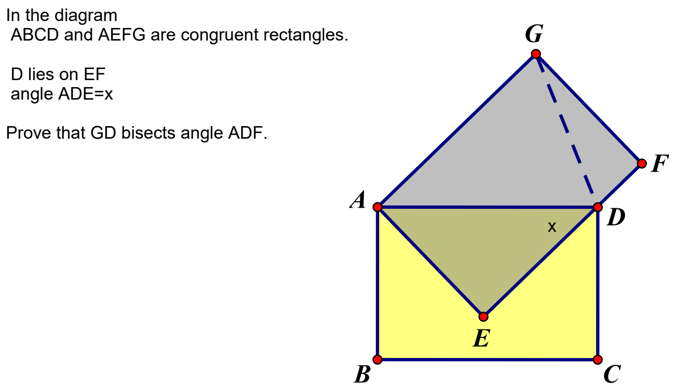

Figure 8. GCSE question on geometric proof (AQA GCSE Mathematics Higher Paper 2 (Calculator), November 2024)

This task is a proving-problem, which expects students to reason deductively to reach a conclusion. However, before proving angle ADG is equal to angle FDG (so that GD is a bisector of angle ADF), it might be useful to 'see' where to start. Look at triangle ADG and deduce it is an isosceles triangle, because the length AD is equal to the length AG. Use alternate angles to determine that angle GAD is also '*x*'. It might be useful to rotate the diagram to 'see' triangle ADG is an isosceles triangle. This is an intuitive skill, which Godfrey (1910) described as 'geometrical eye', rooted in Herbart's views on imagination skills (Herbart, 1802; Treutlein, 1911; Fujita & Jones, 2003; Yamamoto, 2006).

The development of spatial skills should be seen as one of the most essential aims for the teaching of geometry throughout 3-19, so that students are equipped with these skills and can apply them when solving 2D/3D geometry problems as well as in other areas of mathematics and science.

3.4.4 Competencies related to technological tools

Perhaps one of the biggest innovations in the teaching of geometry was the invention of dynamic geometry environments (DGE), which were realised by Cabri (1988), Geometer's Sketchpad (1991), and later GeoGebra (Hohenwarter and Jones, 2007) and Desmos (2007, https://www.desmos.com/). Within a DGE, geometrical objects on a computer screen can be dynamically constructed, manipulated, transformed and measured, enriching discourses related to geometry. We can explore the world of geometry, which might motivate students to learn geometry and appreciate why geometry is needed (Abdelfatah, 2011).

When the 2001 report appeared, DGEs were just being recognised, and it recommended establishing mechanisms to support geometry teaching through ICT (e.g., recommendation 15 d)). Online repositories and shared platforms now provide extensive resources for both teachers and learners.

However, access to and effective use of such tools is still limited, particularly in primary settings. Research studies repeatedly demonstrate that mere use of DGEs cannot support effective teaching and learning, nor develop students' geometrical reasoning and problem-solving skills (Olivero, 1999). DGEs are not the only technological tools, and recently more studies and examples have been explored with, for example, Augmented Reality (AR) or Virtual Reality (VR) (Huang and Fujita, 2025), but we still do not know much about what tasks can be designed and how they can be implemented.

Geraniou and Jankvist (2019) explored competencies related to digital technologies, and identified 'Being aware of which digital tools to apply within different mathematical situations and context, and being aware of the different tools' capabilities and limitations' as one of the competencies. This notion is highly useful when considering the teaching and learning of geometry with different technological tools.

The current research and suggestions for DGEs are rich and diverse, and it is not possible to summarise everything here. Instead, reflecting on our evaluation of the 2001 report and current situation, we quote the following statement made by Keith Jones in 1995, reflecting on a 12-month fieldwork project, in which he used dynamic geometry software with pairs of 11-12-year-old and 13-14-year-old students at a secondary school. He discovered that students found exploring geometry difficult even with dynamic geometry software.

> On reflecting on the problems that I am encountering I should start by emphasising that none of them are due to current poor teaching. On the contrary, the class teacher with whom I am working is not only very experienced but, from the evidence of my observations of classroom practice and from appraisal and inspection reports, the approaches used in the classroom are exemplary. Nevertheless, classroom activities are, to large extent, determined by the school's scheme of work, which is itself determined by the UK National Curriculum. Here is the first problem. Geometry is not well-specified within the UK National Curriculum. (Jones, 1995, p. 32)

Given the current status of geometry in the National Curriculum and teaching programmes, what Jones experienced in 1995 will be widespread by 2026. We need to reconsider the aims of the National Curriculum, design more attractive geometry activities and specify the competencies that should be developed. We also need to consider how technological tools can be integrated into the current classroom environment. Otherwise, we will see a repeat of what Jones experienced in 1995.

Finally, competencies related to geometry discussed in the previous section should be highly useful with the rapidly developing Large Language Model-based generative AI tools. For example, if we put the GCSE proof question in Figure 7 into a generative AI tool, it quickly returned a proof and solution. See Figure 9 for part of the answer:

To prove that GD bisects $\angle ADF$, we need to show that $\angle GDF = \angle GDA$.

Step 1: Analyze the properties of the rectangles

- **Congruency:** Since $ABCD$ and $AEFG$ are congruent rectangles, their corresponding sides and diagonals are equal. Specifically, the diagonals AD and AG are **not** necessarily equal, but we know the side lengths are.
- Let the long side of the rectangles be L ($AB = DC = AE = GF = L$) and the short side be W ($AD = BC = AG = EF = W$).
- Note: In rectangle $AEFG$, the side $AG = EF$. Because D lies on EF, we have $EF = ED + DF$.

Step 2: Use the Right-Angled Triangles

Consider $\triangle AED$ and $\triangle GFD$:

1. **Right Angles:** $\angle AED = 90^\circ$ and $\angle GFD = 90^\circ$ (properties of rectangles).
2. **Equal Sides:** $AE = GF = L$ (congruent rectangles).

Figure 9. Generative AI solution and proof for the GCSE question in Figure 8

However, reading and interpreting such an AI solution requires the above-described competencies, such as identifying triangles AED and GFD. Without these competencies, it is impossible to memorise or understand these lengthy proofs. It is also important to consider whether shorter proofs or chains of reasoning that lead more easily to the desired conclusion might exist. Geometry is a good example of a topic in which students need to exercise both visual and conceptual reasoning, even in the AI era when many areas of mathematics might be automated or answered by AI.

4. Implications (recommendations post-2001 report)

Recommendation 1: Change the curriculum and assessment

1.1: More emphasis on geometrical reasoning including visualisation, spatial skills as well as problem-solving skills

The National Curriculum requires students to become fluent, reason mathematically and spatially, and be able to solve problems. This means designing tasks that allow students to engage in learning that allows them to:

- seek solutions not just memorise procedures
- explore patterns not just memorise formulas
- formulate conjectures, not just do exercises. (Polya, 1945)

One way to make this a guiding principle is to include the use of geometrical reasoning, including visualisation, spatial reasoning, and problem-solving as an explicit requirement for all learners 3-19.

1.2: Use of GDE for geometrical reasoning and competencies

The way teachers teach is a result of their belief system (Hannula, 2016; Honey, 2018; Ernest, 1994). If teachers believe that using technology, and AI in particular, is 'cheating' then they are unlikely to include it as part of their teaching repertoire (Clark-Wilson, 2024). Where teachers are using technology, it is often as a replacement for traditional whiteboards, rather than as a tool for conceptual understanding (Phosa and Mofolo-Mbokane, 2025). Teachers need to be prepared to teach for the students' future and not their own past (Schleicher, 2018).

Crompton et al. (2018) report that technologies such as dynamic geometry environments can support students' 'visualisation, manipulation, cognitive tools, discourse promoters, and new ways of thinking'. Weigand et al. (2025) note that students will have opportunities to develop spatial reasoning using VR and AR environments resulting from gamification. This will require teachers to develop their teaching approaches.

1.3: What do we want to assess?

Testing procedural understanding is straightforward and has been the main way of assessing students' learning. However, this leads to a narrow curriculum, and an unfulfilling learning experience (Whetton, 2009; Hoyles and Healey, 2007).

We recommend the use of multi-part assessment questions, such that factual recall in the first part of a question leads onto geometrical reasoning and problem-solving strategies.

Recommendation 2: Focus on ITE/CPD

2.1: Subject specialists (recruit and retain)

In 2025, the then president of the Mathematical Association, Charlie Stripp, wrote that there is a 'chronic shortage of secondary and post-16 maths teachers, and that this contributes to social inequity and impacts the national economy'. Ofsted (2023) found that poor mathematics provision was often a result of poor recruitment and retention of mathematics specialist teachers, and the use of non-specialist teachers. Often these non-specialist teachers taught in the lower years and in the lower attainment classes; exactly where the need for subject specialists is greatest (Glazzard and Tate, 2025; Sani and Burghes, 2022).

To improve this situation, there needs to be an increase in ITE provision. There also needs to be better support for non-specialist teachers, (given that much of the curriculum is focussed on number and algebra, non-specialist teachers may have had very few opportunities to engage in geometrical reasoning).

2.2: Preparing teachers to use DGE/ICT/AI tools

In order for teachers (whether they are experienced or newly-qualified) to design tasks for geometrical reasoning, and make appropriate use of digital technologies, including AI, they need to have access to high-quality CPD. Without support, teachers are more likely to use digital technologies badly or not at all (Berry et al., 2007; Honey, 2018; Schleicher, 2018).

2.3: Teachers as researchers

Recent government policy changes have led to a shift from university based initial teacher education (ITE) to school-based teacher training (SCITT). The resulting 'apprenticeship model' used by SCITT providers has created a workforce that is mentored by teachers and is based mostly on completing a series of 'tick box exercises' (Pugh et al, 2020). Glazzard and Tate (2025) suggest that the DfE Career Framework (DfE, 2024) '*privileges specific pedagogical approaches, including an emphasis on explicit direct instruction, retrieval, questioning, modelling and explanations. There is no emphasis on problem-based learning and experiential learning*'.

Add to this that teachers teach the way they were taught, and are heavily influenced by their school systems (Ball, 1988; Honey and Graham, 2012; Honey and Graham, 2003), we need to create stronger, longer-lasting links between schools and universities (Oughton et al., 2024).

We recommend that beginning teachers working with SCITT providers should be paired with a school mentor who is actively engaged in practitioner-research, either as part of their own academic development or as part of a research team with the accrediting university. This would provide opportunities for engaging in current research on using geometric reasoning, developing their own understanding of geometry, allowing them to become confident and proficient at designing geometry tasks for their students.

Recommendation 3: funding localised classroom-based research

3.1: Producing rich geometrical tasks, based on sound pedagogy/cultural and historical contexts/modelling real life contexts/enrichment

Much of the school learning experience is based on recall of facts and using procedural content knowledge. This absolutely has a place in the curriculum. However, there needs to be greater emphasis on problem-solving and reasoning. Students should also have access to the historical and cultural background to the 'big ideas' in geometry (Rowlands and Carson, 2006)

Higgins (2005) and Oughton et al. (2024) suggest that teachers need to be supported in structuring activities that engage students and also emphasise the importance of developing teachers' own understanding of the mathematical concepts.

We recommend that teachers engage with high-quality CPD, led by university ITE tutors where they are invited to create classroom tasks that are embedded in sound and appropriate pedagogy. Evidence based practice of this nature will benefit both the teacher and the university tutor.

3.2: Funding more classroom-based research to seek effective ways to develop students' geometrical knowledge and reasoning in the AI era

We recommend more funding for conducting classroom-based research, e.g., how to develop geometrical reasoning, spatial skills and their competencies for the era of AI. Such funding does not have to be huge, but enough for both researchers and teachers to work together to conduct their studies, distributed widely across the UK (and internationally) (Celik, et al., 2022; Sowa et al., 2025; Tripathi, et al., 2025).

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Appendices

Appendix 1 The key principles and recommendations in the 2001 report

Key Principles

Key Principle 1: Geometry should form a significant component of the mathematics curriculum for all students from 11 to 19.

Key Principle 2: Any choice of curriculum should be underpinned by a rationale.

Key Principle 3: The geometry curriculum should maintain breadth, depth and balance, and be consistent with Key Principle 2 and the objectives in Recommendation 3.

Key Principle 4: Geometry should be given a higher status, together with a fair share of the teaching time available for mathematics.

Key Principle 5: Students in 16-19 education should have the opportunity to continue further their studies in geometry.

Key Principle 6: The assessment framework for the curriculum should be designed to ensure that the full range of students' geometrical knowledge, skills and understanding are given credit.

Key Principle 7: The most significant contribution to improvements in geometry teaching will be made by the development of good models of pedagogy, supported by carefully designed activities and resources, which are disseminated effectively and coherently to and by teachers.

Key Principle 8: It is a matter of national importance that as many of our students as possible fully develop their mathematical potential. Geometry, with its distinctive appeal, should make mathematics attractive to a wider range of students.

Recommendations

Recommendation 1: We recommend that curriculum and assessment specifications be reviewed to ensure that geometry forms a significant component of the mathematics curriculum for all students from 11 to 19.

Recommendation 2: We recommend that the title of the attainment target Ma3 of the National Curriculum be changed from 'Shape, space and measures' to 'Geometry'.

Recommendation 3: We recommend that the geometry curriculum be chosen and taught in such a way as to achieve the following objectives:

a) to develop spatial awareness, geometrical intuition and the ability to visualise;

b) to provide a breadth of geometrical experiences in 2 and 3-dimensions;

c) to develop knowledge and understanding of and the ability to use geometrical properties and theorems;

d) to encourage the development and use of conjecture, deductive reasoning and proof;

e) to develop skills of applying geometry through problem-solving and modelling in real world contexts;

f) to develop useful Information & Communication Technology (ICT) skills in specifically geometrical contexts;

g) to engender a positive attitude to mathematics;

h) to develop an awareness of the historical and cultural heritage of geometry in society, and of the contemporary applications of geometry.

Recommendation 4: We recommend that the current geometrical content of the English secondary school mathematics National Curriculum be regarded as a reasonable basis for an appropriate and rewarding geometry education for all learners.

Recommendation 5: We recommend that the mathematics curriculum be developed to encourage students to work investigatively, demonstrate creativity and make discoveries in geometrical contexts so that students develop their powers of spatial thinking, visualisation and geometrical reasoning.

Recommendation 6: We recommend that the mathematics curriculum be developed in ways which recognise the important position of theorems and proofs within mathematics and use the study of geometry to encourage the development of logical argument appropriate to the age and attainment of the student.

Recommendation 7: We recommend that the mathematics curriculum be developed to provide ample opportunities for students to use geometry for practical problem-solving through mathematical modelling in both 2- and 3-dimensions.

Recommendation 8: We recommend that the geometry curriculum be developed to give greater emphasis to work in 3-dimensions and to make better use of Information and Communication Technology (ICT).

Recommendation 9: We recommend that the use of the word 'numeracy' in government publications and announcements be replaced by 'mathematics' to ensure that geometry is accorded its rightful position.

Recommendation 10: We recommend that geometry should occupy 25% - 30% of the teaching time, and hence a similar proportion of the assessment weighting, in the 11-16 mathematics National Curriculum.

Recommendation 11: We recommend that the total time allocated to mathematics 11-16 be monitored to ensure students spend at least 3 hours a week on mathematics, so that sufficient time is given to the teaching of geometry, and to other aspects of mathematics.

Recommendation 12: We recommend that a fundamental review be made of all 16-19 mathematics provision. This should include considering how:

a) the structure and content of the current AS/A-level Mathematics and Further Mathematics specifications can better meet the needs of students and include a greater emphasis on geometry;

b) other post-16 mathematics qualifications, such as Free Standing Mathematics Units (FSMUs) and AS-level Use of Mathematics, can enable students to have the opportunity to continue their study of geometry.

Recommendation 13: We recommend that in the 16-19 curriculum the key skill, 'Application of Number', be retitled 'Application of Mathematics' and that the range of qualifying mathematical studies be broadened so that students continue their study of geometry.

Recommendation 14: We recommend that a review be made of the methods of assessment and examination used in mathematics at Key Stage 3, at GCSE and in post16 qualifications to ensure that appropriate credit is given for the attainment of specific geometrical objectives.

Recommendation 15: We recommend that the relevant government agencies work together, with bodies such as the mathematics professional associations represented on JMC, to provide a coherent framework for supporting the development of teaching and learning in geometry. This will involve:

a) the recognition and development of good practice in geometry teaching through pilot studies and research;

b) the design of programmes of continuing professional development and initial teacher education;

c) the production of supporting materials;

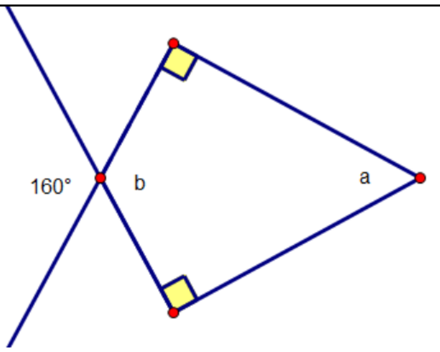
d) the establishment of mechanisms to provide supporting resources, including ICT.

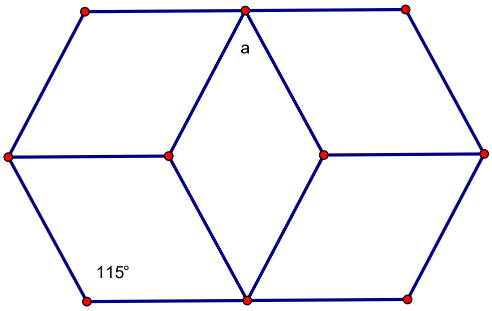
Recommendation 16: We recommend, in terms of mathematics in general, that:

a) better publicity and information be provided to schools, students and parents about the career opportunities afforded by studying mathematics;

b) ways be sought to encourage schools and colleges to attract more students to study mathematics post-16, particularly at A-level.

Appendix 2 Questions used in KS2 tests

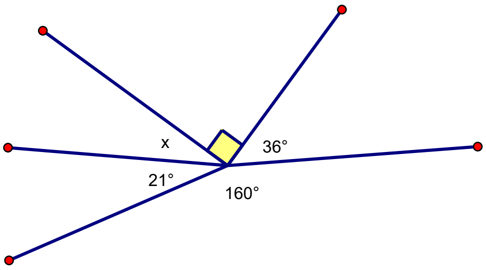
<table>
<tr>
<td>

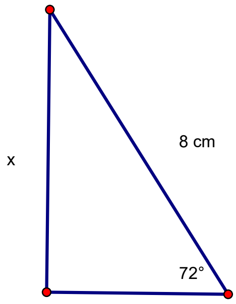
Find a and b.</td>
<td>

The diagram shows four identical parallelograms and a rhombus. Calculate the size of angle a.</td>
</tr>
<tr>
<td>Geometry question on a KS2 Reasoning Paper (2016, Paper 2)</td>
<td>Geometry question on a KS2 Reasoning Paper (2025, Paper 2)</td>
</tr>
</table>

Figure 10. Examples of SATs questions in England

Appendix 3 GSCE questions

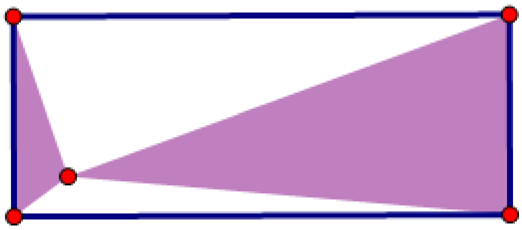
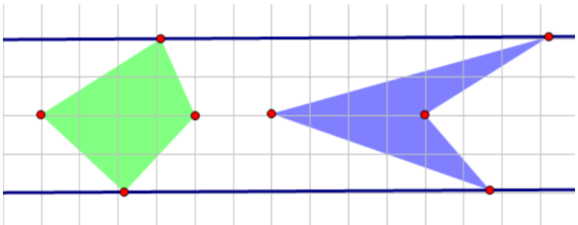
<table>
<tr>
<td>

Find the size of angle x.</td>
<td>

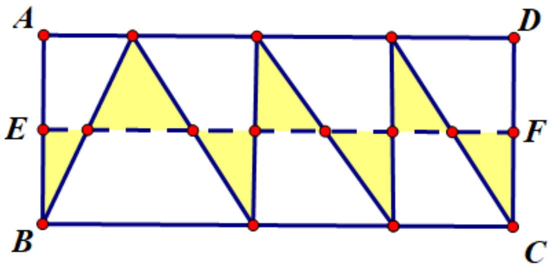
Use trigonometry to work out the length x.</td>
</tr>
<tr>
<td>GCSE Foundation Calculator paper 2017</td>
<td>GCSE Higher Calculator paper 2017</td>
</tr>
</table>

Figure 11. Examples of GCSE questions in England

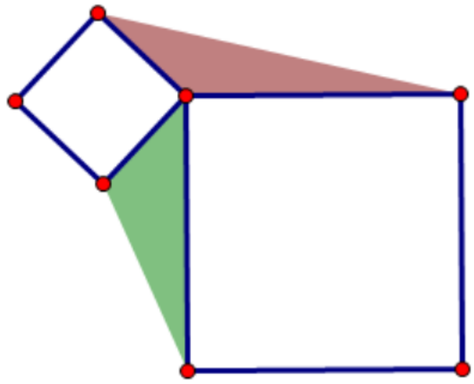
Appendix 4 Tasks for geometrical reasoning

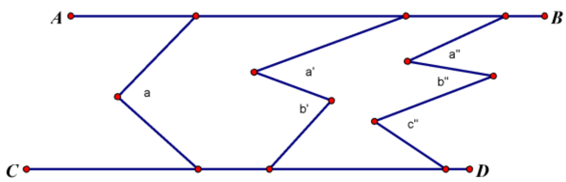
<table>
<tr>
<td>Q. Inside a rectangle, take a point and what can you say the shaded and non-shaded areas? Make your own statement and explore reasons why.
</td>
<td>Q. Two shapes are drawn between parallel lines. What can you say about the areas of these shapes? Make your own statement and explore reasons why.
</td>
</tr>
</table>

Q. ABCD is a rectangle and E and F are midpoints of AB and CD. Points are taken on AD and BC. What can you say about the shaded and non-shaded areas? What would happen if you increase the number of points?

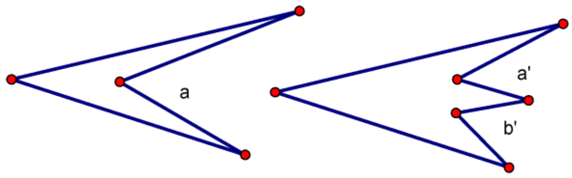


Q. Two squares (with different sizes) are drawn. What relationships can you find about the triangles in between?





AB//CD. What can you say about the angles, a, a' and b', and a'', b'' and c''?



What can you say about the angles a, a' and b' and so on?

Figure 12. Tasks for geometrical reasoning

We recommend that these problems should be explored with pen and paper and a non-digital whiteboard. Why? Because, the 'dynamic, digital environments are for exploring mathematical relationships (conjecturing). For reasoning, it might be more useful to use 'static' environments so that learners can reason why by identifying what mathematical properties can be used (reasoning).

Figure 13 shows a Japanese 12-13-year-old classroom in which the use of digital and non-digital environments are intentionally combined to support conjecturing - reasoning processes in geometry.

Figure 13. Classroom example using digital and non-digital environments together